On matrices of triangular form

(A part of studies on algebraic matric group)

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In this paper we shall introduce the group formed with matrices of triangular form and its subgroups. Then, we think correspondence in their subgroups.

(Definition) Let $\mathfrak{M}$ be a set of non-singular square matrices of degree n. We shall say that $\mathfrak{M}$ is in triangular form, if $a_{ij}=0$ whenever $i>j$, where (a_{ij}) being a matrix in $\mathfrak{M}$. $\mathfrak{M}$ is in special triangular form, if $\mathfrak{M}$ is in triangular form and every matrix in $\mathfrak{M}$ has all the elements on the principal diagonal equal to 1. $\mathfrak{M}$ is in diagonal form, if $a_{ij}=0$ whenever $i \neq j$.

triangular form | special triangular form | diagonal form

$$\left\{\begin{pmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}\right\} \quad \left\{\begin{pmatrix} 1 & & & & * \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}\right\} \quad \left\{\begin{pmatrix} a_{11} & & & & 0 \\ & a_{22} & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}\right\}$$

[Lemma 1] If $\sigma=\begin{pmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}$, then $|\sigma|=a_{11}a_{22}\ldots\ldots a_{nn}$.

[Lemma 2] If $\sigma=\begin{pmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}$ is non-singular matrix, then $a_{11}a_{22}\ldots\ldots a_{nn}\neq 0$

proof is evident.

Theorem 1. If a set $\mathfrak{M}$ is in triangular form, then $\mathfrak{M}$ forms an algebraic matric group.

(proof) Let $\sigma=(a_{ij})$ $\tau=(b_{ij})$ were $a_{ij}=b_{ij}=0$ for $i>j$.

(i) If $\sigma\cdot\tau=(c_{ij})$, then

$$\underset{(i>j)}{c_{ij}}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k<j}a_{ik}b_{kj}+a_{ij}b_{jj}+a_{ij+1}b_{j+1j}+\cdots\cdots+a_{ii}b_{ij}+\sum_{i<k}a_{ik}b_{kj}=0,$$

because a_{ik} $(i>k)$, a_{ij}, b_{j+1j}, $\cdots\cdot$, b_{ij}, b_{kj} $(k>j)$ are all zeros.

(ii) $\sigma^{-1}=\left(\frac{A_{ji}}{\varDelta}\right)=(d_{ij})$, where $\varDelta=|a_{ij}|\neq 0$ and A_{ji} being cofactors of a_{ji}.

$$\underset{(i>j)}{d_{ij}}=\frac{A_{ji}}{\varDelta}=(-1)^{i+j}\frac{1}{\varDelta}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,j-1} & a_{1j} & \cdots & a_{1\,i-1} & a_{1\,i+1} & \cdots & a_{1n} \\ & a_{22} & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ & & & a_{j-1\,j-1} & \vdots & & \vdots & \vdots & & \vdots \\ & & & & 0 & & \vdots & \vdots & & \vdots \\ & 0 & & & & & 0 & \vdots & & \vdots \\ & & & & & & & a_{i+1\,i+1} & & \vdots \\ & & & & & & & & & a_{nn} \end{vmatrix}=0.$$

Therefore σ^{-1} is also in taiangular form.

(iii) $(\sigma\cdot\tau)\cdot\pi=\sigma(\tau\cdot\pi)$

(iv) $E=\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

As above, we can see that $\mathfrak{M}$ forms an algebraic matric group. (q. e. d.)

Theorem 2. If a set $\mathfrak{N}$ is in special triangular form, then $\mathfrak{N}$ forms an normal algebraic subgroup of $\mathfrak{M}$.

(proof) Let $\sigma=(a_{ij})$ $\tau=(b_{ij})$, where $a_{ij}=b_{ij}=0$ for $i>j$
and $a_{ij}=b_{ij}=1$ for $i=j$.

If we put $\sigma\cdot\tau=(c_{ij})$, then by **Theorem 1** $c_{ij}=0$ for $i>j$.

$$c_{ii}=\sum_{k=1}^{n}a_{ik}b_{ki}=\sum_{k>i}a_{ik}b_{ki}+a_{ii}b_{ii}+\sum_{k<i}a_{ik}b_{ki}=1$$

This means that the product is also in special triangular form.

Let $\sigma^{-1}=\left(\frac{A_{ji}}{\varDelta}\right)=(A_{ji})=(d_{ij})$ (for $\varDelta=1$), then obviously $d_{ii}=1$ and by **Theorem 1** $d_{ij}=0$ for $i>j$. Therefore σ^{-1} is in special triangular form, too. Thus $\mathfrak{N}$ forms an algebraic matric group and we can see easily that it is the subgroup of $\mathfrak{M}$.

Put $\sigma_1=(a'_{ij})\in\mathfrak{M}$ $\tau_1=(b'_{ij})\in\mathfrak{N}$ and $\sigma_1\cdot\tau_1=(c'_{ij})$. Then $c_{ii}'=a'_{ii}b'_{ii}=a'_{ii}$.

If $\sigma_1^{-1}=\left(\frac{A'_{ji}}{\varDelta}\right)=(d'_{ij})$ then $d'_{ii}=\frac{1}{a'_{ii}}$. Therefore $\sigma_1\cdot\tau_1\cdot\sigma_1^{-1}\in\mathfrak{N}$, and we see that $\mathfrak{N}$ is normal subgroup of $\mathfrak{M}$. (q. e. d.)

Now we put

$$\mathfrak{T}_1=\left\{\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}\right\}\quad \mathfrak{T}_2=\left\{\begin{pmatrix} a_{11} & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & a_{nn} \end{pmatrix}\right\}\cdots\cdots \mathfrak{T}_k=\left\{\begin{pmatrix} a_{11} & \overbrace{0\cdots0}^{k-1} & & * \\ & \ddots & \ddots & 0 \\ & & \ddots & \vdots \\ 0 & & & \begin{matrix}0\\ a_{nn}\end{matrix} \end{pmatrix}\right\}\cdots\cdots \mathfrak{T}_n=\left\{\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}\right\}$$

$$\mathfrak{S}\mathfrak{T}_1=\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}\quad \mathfrak{S}\mathfrak{T}_2=\left\{\begin{pmatrix} 1 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix}\right\}\cdots\cdots \mathfrak{S}\mathfrak{T}_k=\left\{\begin{pmatrix} 1 & \overbrace{0\cdots0}^{k-1} & & * \\ & \ddots & \ddots & 0 \\ & & \ddots & \vdots \\ 0 & & & \begin{matrix}0\\ 1\end{matrix} \end{pmatrix}\right\}\cdots\cdots \mathfrak{S}\mathfrak{T}_n=\left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$$

[Lemma 3] If $\sigma=(a_{ij})\in\mathfrak{T}_k$, $\tau=(b_{ij})\in\mathfrak{T}_k$ and $\sigma\cdot\tau=(c_{ij})$, then $c_{i\,i+k}=a_{ii}b_{i\,i+k}+a_{i\,i+k}b_{i+k\,i+k}$

(proof) $$c_{i\,i+k}=\sum_{j<i}a_{ij}b_{j\,i+k}+a_{ii}b_{i\,i+k}+a_{i\,i+1}b_{i+1\,i+k}+\cdots+a_{i\,i+k-1}b_{i+k-1\,i+k}+a_{i\,i+k}b_{i+k\,i+k}$$
$$+\sum_{j>i+k}a_{ij}b_{j\,i+k}=a_{ii}b_{i\,i+k}+a_{i\,i+k}b_{i+k\,i+k}$$

(q. e. d.)

[Cor. 1] If $\sigma=(a_{ij})\in\mathfrak{S}\mathfrak{T}_k$, $\tau=(b_{ij})\in\mathfrak{S}\mathfrak{T}_k$ and $\sigma\cdot\tau=(c_{ij})$, then $c_{i\,i+k}=a_{i\,i+k}+b_{i\,i+k}$.

Theorem 3. $\mathfrak{T}_{k+1}$ is the subgroup of $\mathfrak{T}_k$. $\mathfrak{S}\mathfrak{T}_{k+1}$ is the normal subgroup of $\mathfrak{S}\mathfrak{T}_k$.

(proof) we can see that if $\mathfrak{T}_1$, $\mathfrak{T}_2$, ..., $\mathfrak{T}_{k-1}$ form group then $\mathfrak{T}_k$ forms a group, too. Because if we put $\sigma=(a_{ij})$, $\tau=(b_{ij})$ where $a_{ij}=b_{ij}=0$ for $i>j$ and $1<j-i<k$, and $\sigma\cdot\tau=(c_{ij})$ $\frac{A_{ji}}{\triangle}=d_{ij}$, then

(i) $c_{i\,i+k}=0$ (by [Lemma 3])

(ii) $$d_{i\,i+k}=(-1)^k\begin{vmatrix} a_{11} & & & & & & & \\ & a_{22} & & & & & & \\ & & \ddots & & & & & \\ & & & a_{i-1\,i-1} & & & & \\ & & & & 0\cdots\cdots 0 & & & \\ & & & a_{i+1\,i+1} & 0 & \vdots & & \\ & 0 & & \ddots & \ddots & \vdots & & \\ & & & & a_{i+k-1\,i+k-1} & 0 & & \\ & & & & & a_{i+k+1\,i+k+1} & & \\ & & & & & & \ddots & \\ & & & & & & & a_{nn} \end{vmatrix}=(-1)^{2k-1}\begin{vmatrix} a_{11} & & & & & & & \\ & a_{22} & & & & & & \\ & & \ddots & & & & & \\ & & & a_{i-1\,i-1} & & & & \\ & & & & a_{i+1\,i+1} & & & \\ & & & & & \ddots & & \\ & & & & & a_{i+k-1\,i+k-1} & & \\ & 0 & & & & 0 & & \\ & & & & & & a_{i+k+1\,i+k+1} & \\ & & & & & & \ddots & \\ & & & & & & & a_{nn} \end{vmatrix}$$
$$=0$$

by (i) (ii) we find that $\mathfrak{T}_{k+1}$ forms a subgroup of $\mathfrak{T}_k$.

About $\mathfrak{S}\mathfrak{T}_k$, we can see easily that $A_{kk}=1$, $c_{ii}=1$ and then prove similarly that $\mathfrak{S}\mathfrak{T}_{k+1}$ forms a subgroup of $\mathfrak{S}\mathfrak{T}_k$.

Next, if $\sigma_1=(a'_{ij})\in\mathfrak{S}\mathfrak{T}_k$ $\tau_1=(b'_{ij})\in\mathfrak{S}\mathfrak{T}_{k+1}$ $\sigma_1{}^{-1}=(d'_{ij})$ $\sigma_1\cdot\tau_1\cdot\sigma_1{}^{-1}=(e'_{ij})$,

then $$d_{i\,i+k}=A_{i+k\,i}=(-1)^k\begin{vmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0\cdots\cdots a_{i\,i+k} & & & & \\ & & & 1\quad 0 & \vdots & & & \\ & & & \ddots\ \ddots & \vdots & & & \\ & & & 1 & 0 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{vmatrix}=(-1)^{2k+1}\begin{vmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & * & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & a_{i\,i+k} & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{vmatrix}$$
$$=-a_{i\,i+k}.$$

Therefore $e_{i\,i+k}=a_{i\,i+k}+0-a_{i\,i+k}=0$, and then $\sigma\cdot\tau\cdot\sigma_1{}^{-1}\in\mathfrak{S}\mathfrak{T}_{k+1}$.

Thus we can see that $\mathfrak{S}\mathfrak{T}_{k+1}$ forms a normal subgroup of $\mathfrak{S}\mathfrak{T}_k$.

(q. e. d.)

Theorem 4. $\mathfrak{S}\mathfrak{T}_k$ is the normal subgroup of $\mathfrak{T}_k$.

(proof) We can prove with similar method in the proof of **Theorem 2.**

Theorem 5. The factor group of $\mathfrak{T}_k$ by its normal subgroup $\mathfrak{S}\mathfrak{T}_k$ $(k=1, 2, \ldots, n)$ are isomorph each other.

(proof) For instance,

between $\begin{pmatrix} a_{11} & a_{12} & & & & * \\ & a_{22} & a_{23} & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & a_{n-1\,n-1} & a_{n-1\,n} \\ & & & & & a_{nn} \end{pmatrix} \in \mathfrak{S}\mathfrak{T}_1$ and $\begin{pmatrix} a_{11} & 0 & & & & * \\ & a_{22} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & \cdot & & & a_{n-1\,n-1} & 0 \\ & & & & & a_{nn} \end{pmatrix} \in \mathfrak{S}\mathfrak{T}_2$

there exists one to one correspondence. Therefore they are isomorph each other.

(q. e. d.)

Hereafter, this study will be continued until the investigation of original properties of the algebraic matric groups.

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Reference

E. R. Kolchin, Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations. *Ann. of Math,* vol. 49, No. 1 (1948)

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